Trigonometry

1. Simplify the expression: $\sin\left(\frac{\pi}{2}+x\right)+\cos(\pi+x)$

   (A) 0 (B) $2\cos x$ (C) $-2\cos x$ (D) $2\sin x$ (E) NOTA

   Solution: $\sin\left(\frac{\pi}{2}+x\right)+\cos(\pi+x)=\cos x-\cos x=0$

   Answer: (A)

2. If $\sin 2\theta=\frac{7}{9}$ and $0<\theta<\frac{\pi}{2}$, what is $\sin\theta+\cos\theta$?

   (A) $\frac{4}{3}$ (B) $\frac{7}{6}$ (C) $\frac{5}{4}$ (D) $\frac{2}{3}$ (E) NOTA

   Solution: Since $\sin 2\theta=2\sin\theta\cos\theta=\frac{7}{9}$, we have

   $(\sin\theta+\cos\theta)^2=1+2\sin\theta\cos\theta=1+\frac{7}{9}=\frac{16}{9}$. Also, both $\sin\theta$ and $\cos\theta$ are positive, so $\sin\theta+\cos\theta=\frac{4}{3}$.

   Answer: (A)

3. For a given angle θ, find the value of $\cos\theta$ if $\tan\theta=\frac{2}{\sqrt{5}}$ and $\sin\theta<0$.

   (A) $\frac{2}{3}$ (B) $-\frac{2}{3}$ (C) $\frac{\sqrt{5}}{3}$ (D) $-\frac{\sqrt{5}}{3}$ (E) NOTA

   Solution: Since $\tan\theta$ is positive and $\sin\theta$ is negative, θ is an angle in the third quadrant, which produces the cosine value of it negative. Therefore, $\cos\theta=-\frac{\sqrt{5}}{3}$.

   Answer: (D)

4. Which of the following parametric equations represent the elliptic equation

   $$25(x-3)^2+4(y+1)^2=100?$$

   (A) $x=5\cos\theta+3,\ y=2\sin\theta-1$
   (B) $x=5\sin\theta-3,\ y=4\cos\theta+1$
   (C) $x=2\cos\theta+3,\ y=5\sin\theta-1$
   (D) $x=2\cos\theta-3,\ y=5\sin\theta+1$
   (E) NOTA

Solution: $\frac{(x-3)^2}{4}+\frac{(y+1)^2}{25}=1$, so let $\cos\theta=\frac{x-3}{2}$ and $\sin\theta=\frac{y+1}{5}$.

Answer: (C)

5. Which one of the following is positive value when the point $P(-4,5)$ is on the terminal side of angle θ in standard position?

(A) $\sin\theta\cos\theta$ (B) $\csc\theta\tan\theta$ (C) $\tan\theta\sin\theta$ (D)$\sin\theta\sec\theta$
(E) NOTA

Solution: Since the angle θ is in the second quadrant, only $\sin\theta$ and $\csc\theta$ are positive and the others are negative. All values listed above are the products of positive and negative, so they are all negative.

Answer: (E)

6. Evaluate $\sum_{n=1}^{180}\cos n°$.
(A) 0 (B) 1 (C) 2 (D) -1 (E) NOTA

Solution: $\cos(1°)=-\cos(179°)$, $\cos(2°)=-\cos(178°)$, $\cdots$, so

$\sum_{n=1}^{180}\cos n°=\cos(180°)=-1$.

Answer: (D)

7. If $\sin\theta$ and $\cos\theta$ are two roots of an equation $x^2+ax+b=0$ for some angle θ, which of the following has to be always true?
(A) $a^2+2b=-1$
(B) $a^2-2b=1$
(C) $a^2-4b=1$
(D) $a^2+4b=-1$
(E) NOTA

Solution: By Vieta's Formula $\sin\theta+\cos\theta=-a$ and $\sin\theta\cos\theta=b$, so

$a^2-2b=(\sin\theta+\cos\theta)^2-2\sin\theta\cos\theta=1$

Answer: (B)

8. Which one of the following is equal to

$$\arcsin\left(\frac{1}{5}\right)+\arccos\left(\frac{1}{5}\right)+\arctan\left(\frac{1}{5}\right)+\text{arccot}\left(\frac{1}{5}\right)\ ?$$

(A) 0 (B) $\frac{\pi}{2}$ (C) π (D) $\frac{3\pi}{2}$ (E) NOTA

Solution: $\arcsin\left(\frac{1}{5}\right)$ and $\arccos\left(\frac{1}{5}\right)$ are complementary, and so are $\arctan\left(\frac{1}{5}\right)$ and $\text{arccot}(\frac{1}{5})$.

Answer: (C)

9. Find the sum of all roots of the equation $\cos^2 x - \sin x = 1$ where $0 < x < 2\pi$.

(A) $\frac{\pi}{2}$ (B) π (C) 2π (D) $\frac{5\pi}{2}$ (E) NOTA

Solution: The equations is $\sin^2 x + \sin x = 0$. Either $\sin x = 0$ or $\sin x = -1$, so $x = \pi$ or $\frac{3\pi}{2}$.

Answer: (D)

10. Simplify $\arccos\left(\cos\frac{5\pi}{4}\right)$.

(A) $\frac{\pi}{4}$ (B) $\frac{3\pi}{4}$ (C) $\frac{5\pi}{4}$ (D) $-\frac{\pi}{4}$ (E) NOTA

Solution: $\arccos\left(\cos\frac{5\pi}{4}\right) = \arccos\left(-\frac{1}{\sqrt{2}}\right) = \frac{3\pi}{4}$.

Answer: (B)

11. Which one of the following trigonometric expression is identical to $\cos x \cdot (\sec x - \cos x)$?

(A) $\cos^2 x$
(B) $\sin^2 x$
(C) $\tan^2 x$
(D) $\sin x \cos x$
(E) NOTA

Solution: $\cos x\,(\sec x - \cos x) = \cos\left(\frac{1}{\cos x} - \cos x\right) = 1 - \cos^2 x = \sin^2 x$

Answer: (B)

12. When $\cos\theta = -\frac{5}{13}$, what is the value of $\cos 2\theta$?

(A) $\frac{25}{169}$ (B) $-\frac{50}{169}$ (C) $\frac{144}{169}$ (D) $-\frac{119}{169}$ (E) NOTA

Solution: $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 2\left(-\frac{5}{13}\right)^2 - 1 = -\frac{119}{169}$

Answer: (D)

13. What is the value of $\sin\left(2\arcsin\frac{1}{3}\right)$?

(A) $\frac{2}{3}$ (B) $\frac{4\sqrt{2}}{3}$ (C) $\frac{4\sqrt{2}}{9}$ (D) $\frac{2}{9}$ (E) NOTA

Solution: Let $\theta = \arcsin\frac{1}{3}$, then $\sin\theta = \frac{1}{3}$, and hence $\cos\theta = \frac{2\sqrt{2}}{3}$.

Now $\sin 2\theta = 2\sin\theta\cos\theta = 2\cdot\frac{1}{3}\cdot\frac{2\sqrt{2}}{3} = \frac{4\sqrt{2}}{9}$

Answer: (C)

14. Given that $\sin x - \sin y = \frac{4}{5}$ and $\cos x + \cos y = \frac{3}{5}$, find $\cos(x+y)$.

(A) $\frac{3}{4}$ (B) $-\frac{3}{4}$ (C) $-\frac{1}{2}$ (D) $\frac{1}{2}$ (E) NOTA

Solution:$(\sin x - \sin y)^2 = 1 - 2\sin x\sin y = \frac{16}{25}$,

and $(\cos x + \cos y)^2 = 1 + 2\cos x\cos y = \frac{9}{25}$.

By combining the two equations $\cos(x+y) = \cos x\cos y - \sin x\sin y = -\frac{1}{2}$.

Answer: (C)

15. Which of the following angles should satisfy the inequality

$$2^{cosx} + 2^{sinx} < 2^{cosx+sinx} + 1 \text{ ?}$$

(A) $36°$ B) $110°$ C) $292°$ D) $310°$ E) NOTA

Solution: The inequality yields $2^{\cos x}\cdot 2^{\sin x} - 2^{\cos x} - 2^{\sin x} + 1 > 0$ which is equivalent to $(2^{\cos x} - 1)(2^{\sin x} - 1) > 0$. Thus, either $\cos x > 0$ and $\sin x > 0$, or $\cos x < 0$ and $\sin x < 0$. Finding an angle located in the first or the third quadrant, the answer is (A).

Answer: (A)

16. Let $f(x) = \sin x$ and $g(x) = \cos x$ be two functions defined on $[0, \frac{\pi}{2}]$. Which of the four functions, $f(f(x)), f(g(x)), g(f(x)), g(g(x))$, are increasing over $[0, \frac{\pi}{2}]$?

(A) $f(g(x))$ and $g(f(x))$
(B) $f(f(x))$ and $g(g(x))$
(C) $f(g(x))$ and $g(g(x))$
(D) $f(f(x))$ and $g(f(x))$

(E) NOTA

Solution: Note that $f(x)$ is increasing on $[0, \frac{\pi}{2}]$, and $g(x)$ is decreasing on $[0, \frac{\pi}{2}]$ where ranges of both functions lie in $[0, \frac{\pi}{2}]$. Then the composition of an increasing function with an increasing function or the composition of a decreasing function with a decreasing function yields an increasing function on the given interval.

Answer: (B)

17. If $\tan\theta + \cot\theta = 5$, what is the value of $\csc^2\theta + \sec^2\theta$?

(A) 2 (B) 5 (C) 23 (D) 25 (E) NOTA

Solution: $\tan^2\theta + 2 + \cot^2\theta = \sec^2\theta + \csc^2\theta = 25$

Answer: (D)

18. When the solution set of the equation

$$\lfloor \sin x \rfloor + \lfloor 2\sin x \rfloor + \lfloor 3\sin x \rfloor = 1 \text{ for } x \text{ in } \left[0, \frac{\pi}{2}\right] \text{ is written as } \alpha \le x < \beta,$$

what is $\cos(\alpha + \beta)$?

(A) $\frac{\sqrt{6}}{3} + \frac{1}{6}$ (B) $\frac{\sqrt{6}}{3} - \frac{1}{6}$ (C) $\frac{\sqrt{6}}{6} + \frac{1}{3}$ (D) $\frac{\sqrt{6}}{6} - \frac{1}{3}$ (E) NOTA

Solution: Note that $3\sin x \ge 1$ and $2\sin x < 1$. So $\frac{1}{3} \le \sin x < \frac{1}{2}$ or equivalently,

$\arcsin\frac{1}{3} \le x < \frac{\pi}{6}$. Therefore $\alpha = \arcsin\frac{1}{3}$ and $\beta = \frac{\pi}{6}$.

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta = \frac{2\sqrt{2}}{3} \cdot \frac{\sqrt{3}}{2} - \frac{1}{3} \cdot \frac{1}{2} = \frac{\sqrt{6}}{3} - \frac{1}{6}.$$

Answer: (B)

19. Which of the following intervals can be a domain of the function $f(x) = \frac{1}{\sqrt{1 - 4\sin^2 x}}$?

(A) $-\frac{\pi}{3} < x < \frac{\pi}{3}$ (B) $\frac{\pi}{3} < x < \frac{2\pi}{3}$ (C) $\frac{\pi}{6} < x < \frac{5\pi}{6}$ (D) $\frac{5\pi}{6} < x < \frac{7\pi}{6}$ (E) NOTA

Solution: $1 - 4\sin^2 x > 0$, so $-\frac{1}{2} < \sin x < \frac{1}{2}$. Solving the triangular inequality, we obtain solution sets like $-\frac{\pi}{6} < x < \frac{\pi}{6}$ or $\frac{5\pi}{6} < x < \frac{7\pi}{6}$.

Answer: (D)

20. Which pair of the following graphs coincide?

a) $y = 3\sin 2\left(x - \frac{\pi}{4}\right)$
b) $y = -3\sin 2x$
c) $y = -3\cos 2x$
d) $y = 3\cos 2\left(x - \frac{\pi}{4}\right)$

(A) a and b (B) b and c (C) c and d (D) a and c (E) NOTA

Solution: $3\sin 2\left(x - \frac{\pi}{4}\right) = 3\sin\left(2x - \frac{\pi}{2}\right) = 3\sin(2x)\cos\left(\frac{\pi}{2}\right) - 3\cos(2x)\sin\left(\frac{\pi}{2}\right) = -3\cos(2x)$

Answer: (D)

21. Four points A, B, C, D lie on the circumference of a circle to form a quadrilateral. Let $\alpha, \beta, \gamma, \delta$ denote four interior angles of the quadrilateral associated with A, B, C, D, respectively. Which of the following is NOT true?

(A) $\cos\beta\cos\delta = \sin\beta\sin\delta + 1$
(B) $\sin\alpha\cos\gamma + \cos\alpha\sin\gamma = 0$
(C) $\sin^2\alpha + \cos^2\gamma = 1$
(D) $\cos\beta + \cos\delta = 0$
(E) NOTA

Solution: Note that $\alpha + \gamma = \pi$ and $\beta + \delta = \pi$. (C) and (D) follow from the fact immediately. And $\sin\alpha\cos\gamma + \cos\alpha\sin\gamma = \sin(\alpha+\gamma) = \sin\pi = 0$ which shows (B). However, $\cos\beta\cos\delta - \sin\beta\sin\delta = \cos(\beta+\delta) = \cos\pi = -1$.

Answer: (A)

22. What is $\cot 80° \cot 55° + \cot 80° + \cot 55°$?

(A) 0 (B) 1 (C) 2 (D) 3 (E) NOTA

Solution: Since $\tan 135° = \frac{\tan 55° + \tan 80°}{1 - \tan 55° \tan 80°} = -1$,

$\tan 55° + \tan 80° = -1 + \tan 55° \tan 80°$. Now,

$$\cot 80° \cot 55° + \cot 80° + \cot 55° = \frac{1}{\tan 80° \tan 55°} + \frac{1}{\tan 80°} + \frac{1}{\tan 55°}$$
$$= \frac{1 + \tan 80° + \tan 55°}{\tan 80° \tan 55°} = \frac{1 + (-1 + \tan 80° \tan 55°)}{\tan 80° \tan 55°} = 1$$

Answer: (B)

23. Let a_n be a sequence which represents the number of intersecting points of two graphs, $y = \sin x$ and $y = \cos 2nx$, over the open interval $(0,\ 2\pi)$. Write the general term of the sequence a_n.
(A) $2n$ (B) $4n$ (C) $4n - 1$ (D) $8n - 5$ (E) NOTA

Solution: The period of the graph of $y = \cos 2nx$ is $\frac{\pi}{n}$, so there are $2n$ complete cycle of cosine graphs between 0 and 2π. Two graphs of $y = \sin x$ and $y = \cos 2nx$ are tangent either at $x = \frac{\pi}{2}$ when n is even or at $x = \frac{3\pi}{2}$ when n is odd. Thus, the number of intersections is $2(2n) - 1$.

Answer: (C)

24. Which of the following is equal to the infinite sum
$\sin x + \sin x \cos^2 x + \sin x \cos^4 x + \sin x \cos^6 x + \cdots$ for x in $(0, \pi)$?
(A) $\sin x$ (B) $\csc x$ (C) $\cos x$ (D) $\cos x$ (E) NOTA

Solution:

$$\sin x + \sin x \cos^2 x + \sin x \cos^4 x + \sin x \cos^6 x + \cdots = \frac{\sin x}{1 - \cos^2 x} = \frac{1}{\sin x} = \csc x$$

Answer: (B)

25. How many solutions to the equation $\cos^2 x - 3\cos x - 4 = 0$ are there on the open interval $(0,\ 2\pi)$?
(A) 1 (B) 2 (C) 3 (D) 4 (E) NOTA

Solution: Since $\cos^2 x - 3\cos x - 4 = (\cos x - 4)(\cos x + 1) = 0$ and $-1 \le \cos x \le 1$ for all x, we have $\cos x = -1$, and hence there is only one root , $x = \pi$ on the interval $(0,\ 2\pi)$.

Answer: (A)

26. Aaron and Bill watch a drone flying 120 feet above the ground. The angle of the elevation from Aaron to the drone is $45°$ and from Bill to the drone is $60°$. Assuming that the

positions of Aaron and Bill and the point of perpendicular projection from the drone to the ground form a line, how far are Aaron and Bill apart?

(A) $120 - 40\sqrt{3}$ (B) $120 - 120\sqrt{3}$ (C) $120 + \sqrt{3}$

(D) $120\sqrt{3}$ (E) NOTA

Solution: Let x be the distance from Aaron to Bill and let y be the distance from Bill to the projection point of the drone on the ground. Then $x + y = 120$ and $y = \frac{120}{\tan 60°} = 40\sqrt{3}$, and hence $x = 120 - 40\sqrt{3}$

Answer: (A)

27. Let F_n be the sequence with $F_1 = F_2 = 1, F_{n+2} = F_{n+1} + F_n$. Define a sequence, z_n, of complex numbers by $z_n = \cos F_n + i \sin F_n$. Which of the following is true for z_n ?
(A) $z_{n+2} = z_{n+1} + z_n$
(B) $z_{n+1} = 2z_n$
(C) $z_{n+2} = z_{n+1}z_n$
(D) $z_n^2 = z_{2n}$
(E) NOTA

Solution:

$$z_{n+2} = \cos F_{n+2} + i \sin F_{n+2} = \cos(F_{n+1} + F_n) + i \sin(F_{n+1} + F_n)$$
$$= (\cos F_{n+1} + i \sin F_{n+1})(\cos F_n + i \sin F_n) = z_{n+1}z_n$$

Answer: (C)

28. Let G_n be the sequence with $G_1 = 1, G_{n+1} = 2G_n$. Define a sequence, w_n, of complex numbers by $w_n = \cos G_n + i \sin G_n$. Which of the following is true for w_n ?
(A) $w_{n+2} = w_{n+1} + w_n$
(B) $w_{n+1} = 2w_n$
(C) $w_{n+2} = w_{n+1}w_n$
(D) $w_n^2 = w_{2n}$
(E) NOTA

Solution: $w_n^2 = (\cos G_n + i \sin G_n)^2 = \cos 2G_n + i \sin 2G_n = w_{n+1}$

Answer: (D)

29. Which of the following is equal to $\cos\frac{2\pi}{5}$?

(A) $\frac{\sqrt{5}+1}{4}$ (B) $\frac{\sqrt{5}-1}{4}$ (C) $\frac{\sqrt{6}+\sqrt{2}}{4}$ (D) $\frac{\sqrt{6}-\sqrt{2}}{4}$ (E) NOTA

(Solution) Let $\theta = \frac{2\pi}{5}$, then $5\theta = 2\pi$. Since $\sin 3\theta = \sin(2\pi - 2\theta) = \sin 2\theta$ and $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$, we have $2\sin\theta\cos\theta = 3\sin\theta - 4\sin^3\theta$, and so $3 - 4\sin^2\theta = 2\cos\theta$ which yields a quadratic equation $4\cos^2\theta - 2\cos\theta - 1 = 0$. Solving this equation for $\cos\theta$, we obtain the positive value of $\cos\theta = \frac{-1+\sqrt{5}}{4}$.

Answer: (B)

30. Evaluate the product: $\sin 10° \sin 30° \sin 50° \sin 70°$

(A) $\frac{1}{8}$ (B) $\frac{1}{16}$ (C) $\frac{1}{32}$ (D) $\frac{1}{64}$ (E) NOTA

Solution: $\sin 10° \sin 70° \sin 30° \sin 50°$

$$= \frac{\sin 10° \sin 20° \sin 30° \sin 40° \sin 50° \sin 60° \sin 70° \sin 80°}{\sin 20° \sin 40° \sin 60° \sin 80°}$$

$$= \frac{\sin 10° \sin 20° \sin 30° \sin 40° \cos 40° \cos 30° \cos 20° \cos 10°}{2\sin 10° \cos 10° \cdot 2\sin 20° \cos 20° \cdot 2\sin 30° \cos 30° \cdot 2\sin 40° \cos 40°}$$

$$= \frac{1}{16}$$

Answer: (B)